

# On $(n, k)$ -quasi- $*$ -paranormal operators <sup>\*</sup>

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**Abstract:** For nonnegative integers  $n$  and  $k$ , we introduce in this paper a new class of  $(n, k)$ -quasi- $*$ -paranormal operators satisfying

$$\|T^{1+n}(T^k x)\|^{1/(1+n)} \|T^k x\|^{n/(1+n)} \geq \|T^*(T^k x)\| \text{ for all } x \in H.$$

This class includes the class of  $n$ - $*$ -paranormal operators and the class of  $(1, k)$ -quasi- $*$ -paranormal operators contains the class of  $k$ -quasi- $*$ -class  $A$  operators. We study basic properties of  $(n, k)$ -quasi- $*$ -paranormal operators: (1) inclusion relations and examples; (2) a matrix representation; (3) joint (approximate) point spectrum and single valued extension property.

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## 1 Introduction

Let  $L(H)$  stand for the  $C^*$  algebra of all bounded linear operators on an infinite dimensional complex Hilbert space  $H$ . As an extension of normal operators, P. Halmos introduced the class of hyponormal operators (defined by  $TT^* \leq T^*T$ ) [12]. Although there are still many interesting problems for hyponormal operators yet to solve (e.g., the invariant subspace problem), one of recent hot topics in operator theory is to study natural extensions of hyponormal operators. Below are some of these nonhyponormal operators. Recall that an operator  $T \in L(H)$  is said to be

- $*$ -class  $A$  if  $|T^2| \geq |T^*|^2$  (see [10]).
- quasi- $*$ -class  $A$  if  $T^*|T^2|T \geq T^*|T^*|^2T$  (see [21]).
- $k$ -quasi- $*$ -class  $A$  if  $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$  (see [18]).
- $*$ -paranormal if  $\|T^2x\|^{1/2}\|x\|^{1/2} \geq \|T^*x\|$  for all  $x \in H$  (see [5]).
- quasi- $*$ -paranormal if  $\|T^2(Tx)\|^{1/2}\|Tx\|^{1/2} \geq \|T^*(Tx)\|$  for all  $x \in H$  (see [19]).
- $k$ -quasi- $*$ -paranormal if  $\|T^2(T^kx)\|^{1/2}\|T^kx\|^{1/2} \geq \|T^*(T^kx)\|$  for all  $x \in H$ .
- $n$ - $*$ -paranormal if  $\|T^{1+n}x\|^{1/(1+n)}\|x\|^{n/(1+n)} \geq \|T^*x\|$  for all  $x \in H$  (see [16]).

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Here and henceforth,  $n, k$  denote nonnegative integers.

Clearly, if  $k = 1$  (respectively,  $k = 0$ ), then  $k$ -quasi- $*$ -class  $A$  is precisely quasi- $*$ -class  $A$  (respectively,  $*$ -class  $A$ ), and  $k$ -quasi- $*$ -paranormal is precisely quasi- $*$ -paranormal (respectively,  $*$ -paranormal); if  $n = 1$  (respectively,  $n = 0$ ), then  $n$ - $*$ -paranormal is precisely  $*$ -paranormal (respectively, hyponormal). Moreover,  $*$ -class  $A$  operators are  $*$ -paranormal [10, Theorem1.3], and  $k$ -quasi- $*$ -class  $A$  operators are  $k$ -quasi- $*$ -paranormal (see Theorem 2.1 below) and contain  $*$ -class  $A$ .

As an extension of the classes of  $n$ - $*$ -paranormal operators and  $k$ -quasi- $*$ -paranormal operators, the following definition describes the class of operators we will study in this paper.

**Definition 1.1.** *An operator  $T \in L(H)$  is said to be  $(n, k)$ -quasi- $*$ -paranormal if*

$$||T^{1+n}(T^k x)||^{1/(1+n)} ||T^k x||^{n/(1+n)} \geq ||T^*(T^k x)|| \text{ for all } x \in H.$$

Clearly, if  $n = 1$  (respectively,  $n = 0$ ), then  $(n, k)$ -quasi- $*$ -paranormal is precisely  $k$ -quasi- $*$ -paranormal (respectively,  $k$ -quasihyponormal which is introduced in [8] and defined by  $T^{*k} T T^* T^k \leq T^{*k} T^* T T^k$ ); if  $k = 0$ , then  $(n, k)$ -quasi- $*$ -paranormal is precisely  $n$ - $*$ -paranormal.

In this study we give basic properties of  $(n, k)$ -quasi- $*$ -paranormal operators. In Section 2, we discuss some inclusion relations and examples related to  $(n, k)$ -quasi- $*$ -paranormal operators. In Section 3, a matrix representation is obtained and a negative answer to a question posed by Mecheri [18] is given. In Section 4, we show that, for every  $(n, k)$ -quasi- $*$ -paranormal operator  $T$ , the nonzero points of its point spectrum and joint point spectrum are identical, the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical. As a corollary, it is also shown that  $(n, k)$ -quasi- $*$ -paranormal operators have the single valued extension property.

## 2 Inclusion relations and examples

Recall that an operator  $T \in L(H)$  is said to be

- normaloid if  $||T|| = r(T)$ , where  $r(T)$  is the spectral radius of  $T$  (see [11]).
- hereditarily normaloid if every part of  $T$  is normaloid, where a part of  $T$  means its restriction to a closed invariant subspace (see [9]).
- $n$ -paranormal if  $||T^{1+n} x||^{1/(1+n)} ||x||^{n/(1+n)} \geq ||Tx||$  for all  $x \in H$  (see [13]).
- $(n, k)$ -quasiparanormal if  $||T^{1+n}(T^k x)||^{1/(1+n)} ||T^k x||^{n/(1+n)} \geq ||T(T^k x)||$  for all  $x \in H$  (see [27]).

Clearly,  $(n, 0)$ -quasiparanormal is precisely  $n$ -paranormal.

**Theorem 2.1.** *The following assertions hold.*

- (1) *If  $T$  is  $k$ -quasi- $*$ -class  $A$ , then it is  $k$ -quasi- $*$ -paranormal.*
- (2) *If  $T$  is  $(n, k + 1)$ -quasi- $*$ -paranormal, then it is  $(n + 1, k)$ -quasiparanormal.*
- (3) *If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal and  $M$  is an invariant subspace of  $T$ , then  $T|_M$  is also  $(n, k)$ -quasi- $*$ -paranormal.*
- (4) *If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal, then it is  $(n, k + 1)$ -quasi- $*$ -paranormal.*

(5) If  $T$  is  $(n, 0)$ -quasi- $*$ -paranormal or  $(n, 1)$ -quasi- $*$ -paranormal, then it is hereditarily normaloid.

Theorem 2.1(5) generalizes [18, Theorem 2.6]. On the other hand, for  $k \geq 2$ , there exists an  $(n, k)$ -quasi- $*$ -paranormal but not normaloid operator (see Example 2.3(4) below).

*Proof.* (1) For all  $x \in H$ , we have

$$(T^{*k}|T^*|^2T^kx, x) = (T^{*k}TT^*T^kx, x) = (T^*T^kx, T^*T^kx) = \|T^*T^kx\|^2$$

and by Hölder-McCarthy inequality [17, Lemma 2.1],

$$\begin{aligned} (T^{*k}|T^2|T^kx, x) &= (|T^2|T^kx, T^kx) \\ &\leq (T^{*2}T^2T^kx, T^kx)^{1/2} \|T^kx\|^{2(1-1/2)} \\ &= \|T^{k+2}x\| \|T^kx\|. \end{aligned}$$

Since  $T$  is  $k$ -quasi- $*$ -class  $A$ ,  $\|T^{k+2}x\| \|T^kx\| \geq \|T^*(T^kx)\|^2$ , for all  $x \in H$ . Hence  $T$  is  $k$ -quasi- $*$ -paranormal.

(2) Since  $T$  is  $(n, k+1)$ -quasi- $*$ -paranormal, we have, for all  $x \in H$ ,

$$\begin{aligned} \|T^{k+1}x\|^{2n+2} &= (T^{k+1}x, T^{k+1}x)^{n+1} \\ &= (T^*T^{k+1}x, T^kx)^{n+1} \\ &\leq \|T^*T^{k+1}x\|^{n+1} \|T^kx\|^{n+1} \\ &\leq \|T^{1+n}T^{k+1}x\| \|T^{k+1}x\|^n \|T^kx\|^{n+1}. \end{aligned}$$

Therefore,  $\|T(T^kx)\|^{n+2} \leq \|T^{n+2}(T^kx)\| \|T^kx\|^{n+1}$ , for all  $x \in H$ . Hence  $T$  is  $(n+1, k)$ -quasiparanormal.

(3) It is clear.

(4) It follows by taking  $x = Tz$  in the definition.

(5) By (3) and (4), it needs only to show that  $T$  is normaloid when  $T$  is  $(n, 1)$ -quasi- $*$ -paranormal. By (2), we have that  $T$  is  $(n+1, 0)$ -quasiparanormal, that is,  $T$  is  $(n+1)$ -paranormal. It then follows from [13, Theorem 1] that  $T$  is normaloid.  $\square$

In order to establish the proper inclusion relation among the class of  $(n, k)$ -quasi- $*$ -paranormal operators and that of  $(n, k+1)$ -quasiparanormal operators, the following lemma is needful.

**Lemma 2.2.** *An operator  $T \in L(H)$  is  $(n, k)$ -quasi- $*$ -paranormal if and only if*

$$T^{*k}T^{*(1+n)}T^{1+n}T^k - (1+n)\mu^n T^{*k}TT^*T^k + n\mu^{1+n}T^{*k}T^k \geq 0 \quad (2.1)$$

for all  $\mu > 0$ .

This is a generalization of [20, Theorem 4.14].

*Proof.* The proof is similar to [27, Lemma 2.2] and [26, Lemma 2.2]. Let  $T$  be  $(n, k)$ -quasi- $*$ -paranormal. It then follows from the weighted arithmetic-geometric mean inequality that

$$\begin{aligned} & \frac{1}{1+n}(\mu^{-n}|T^{1+n}|^2T^kx, T^kx) + \frac{n}{1+n}(\mu T^kx, T^kx) \\ & \geq (\mu^{-n}|T^{1+n}|^2T^kx, T^kx)^{\frac{1}{1+n}}(\mu T^kx, T^kx)^{\frac{n}{1+n}} \\ & = (|T^{1+n}|^2T^kx, T^kx)^{\frac{1}{1+n}}(T^kx, T^kx)^{\frac{n}{1+n}} \\ & \geq (|T^*|^2T^kx, T^kx) = (TT^*T^kx, T^kx). \end{aligned}$$

Conversely, by (2.1), we have

$$(T^{*k}T^{*(1+n)}T^{1+n}T^kx, x) - (1+n)\mu^n(T^{*k}TT^*T^kx, x) + n\mu^{1+n}(T^{*k}T^kx, x) \geq 0 \quad (2.2)$$

for all  $x \in H$ . If  $|T^{1+n}|^2T^kx, T^kx = 0$ , multiplying (2.2) by  $\mu^{-n}$  and letting  $\mu \rightarrow 0$  we have  $(TT^*T^kx, T^kx) = 0$ , thus

$$\|T^{1+n}(T^kx)\| \|T^kx\|^n \geq \|T^*(T^kx)\|^{1+n}.$$

If  $|T^{1+n}|^2T^kx, T^kx > 0$ , putting

$$\mu = \left( \frac{(|T^{1+n}|^2T^kx, T^kx)}{(T^kx, T^kx)} \right)^{\frac{1}{1+n}}$$

in (2.2) we have

$$(|T^{1+n}|^2T^kx, T^kx)^{1/(1+n)}(T^kx, T^kx)^{n/(1+n)} \geq (TT^*T^kx, T^kx).$$

So,  $T$  is  $(n, k)$ -quasi- $*$ -paranormal. □

To illustrate the result established in Theorem 2.1, we consider the following

**Example 2.3.** (1) An example of  $*$ -paranormal but not  $*$ -class  $A$  operator.

Similar to an argument of Ando [4], Duggal et al. showed in [10, p960] that there exists a  $*$ -paranormal operator  $T \in L(K)$  such that  $T \otimes T$  is not  $*$ -paranormal:

$$T := T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \text{ on } K = \bigoplus_{n=1}^{\infty} H,$$

where  $\dim H = 2$ ,  $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{1/2}$  and  $B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}^{1/4}$ . From [10, Theorem 3.2], it follows that  $T$  is not  $*$ -class  $A$ . Otherwise,  $T \otimes T$  is  $*$ -class  $A$ , which is impossible.

(2) An example of  $(n+1, k)$ -quasiparanormal but not  $(n, k+1)$ -quasi- $*$ -paranormal operator, for nonnegative integers  $n, k$ .

Let  $U$  be the unilateral right shift operator on  $l_2(\mathbb{N})$  with the canonical orthogonal basis  $\{e_m\}_{m=1}^\infty$  defined by  $Ue_m = e_{m+1}$  for all  $m \in \mathbb{N}$ . Put

$$T = \begin{pmatrix} 1 & \alpha e_1 \otimes e_1 \\ 0 & U + 1 \end{pmatrix} \text{ on } H = \mathbb{C}e_1 \oplus l_2(\mathbb{N}).$$

Uchiyama proved in [23] that  $T$  is paranormal for  $0 < \alpha < \frac{1}{4}$  and  $\ker(T-1) = \mathbb{C}e_1 \oplus \{0\}$  does not reduce  $T$ . By [27, Theorem 2.1], we have that  $T$  is  $(n+1, k)$ -quasiparanormal for all nonnegative integers  $n, k$ . But, by Theorem 4.1 below,  $T$  is not  $(n, k+1)$ -quasi- $*$ -paranormal for all nonnegative integers  $n, k$ , because  $\ker(T-1)$  does not reduce  $T$ .

(3) An example of  $(n, k+1)$ -quasi- $*$ -paranormal but not  $(n, k)$ -quasi- $*$ -paranormal operator, for nonnegative integer  $n$  and positive integer  $k \geq 1$ .

Given a wight sequence  $\{w_m\}_{m=1}^\infty$  of bounded and positive numbers, let  $T$  be the unilateral weighted right shift operator on  $l_2(\mathbb{N})$  with the canonical orthogonal basis  $\{e_m\}_{m=1}^\infty$  defined by  $Te_m = w_m e_{m+1}$  for all  $m \in \mathbb{N}$ . A routine calculation show that (see [27, Example 2.3]),  $TT^* = 0 \oplus w_1^2 \oplus w_2^2 \oplus \dots$ , and for each positive integer  $m$ ,

$$T^{*m}T^m = w_1^2 \cdots w_m^2 \oplus w_2^2 \cdots w_{m+1}^2 \oplus w_3^2 \cdots w_{m+2}^2 \oplus \dots \text{ on } l_2(\mathbb{N}), \quad (2.3)$$

$$T^{*m}TT^*T^m = w_1^2 \cdots w_m^2 w_m^2 \oplus w_2^2 \cdots w_{m+1}^2 w_{m+1}^2 \oplus w_3^2 \cdots w_{m+2}^2 w_{m+2}^2 \oplus \dots \text{ on } l_2(\mathbb{N}). \quad (2.4)$$

By (2.3), (2.4) and Lemma 2.2,  $T$  is  $(n, k)$ -quasi- $*$ -paranormal if and only if, for all  $\mu > 0$ ,

$$\begin{cases} w_{k+1}^2 \cdots w_{k+n+1}^2 - (n+1)\mu^n w_k^2 + n\mu^{n+1} \geq 0, \\ w_{k+2}^2 \cdots w_{k+n+2}^2 - (n+1)\mu^n w_{k+1}^2 + n\mu^{n+1} \geq 0, \\ w_{k+3}^2 \cdots w_{k+n+3}^2 - (n+1)\mu^n w_{k+2}^2 + n\mu^{n+1} \geq 0, \\ \dots\dots\dots \end{cases}$$

It then follows that  $T$  is  $(n, k)$ -quasi- $*$ -paranormal if and only if

$$\begin{cases} w_k^{n+1} \leq w_{k+1} \cdots w_{k+n+1}, \\ w_{k+1}^{n+1} \leq w_{k+2} \cdots w_{k+n+2}, \\ w_{k+2}^{n+1} \leq w_{k+3} \cdots w_{k+n+3}, \\ \dots\dots\dots \end{cases} \quad (2.5)$$

So, if  $w_{k+1} \leq w_{k+2} \leq w_{k+3} \leq \dots$  and  $w_k > w_{k+n+1}$ , then  $T$  is  $(n, k+1)$ -quasi- $*$ -paranormal but not  $(n, k)$ -quasi- $*$ -paranormal.

(4) An example of  $(n, k)$ -quasi- $*$ -paranormal but not normaloid operator, for  $k \geq 2$ .

Let  $T$  be the unilateral weighted right shift operator as in (3) and let

$$w_1 > w_2 = \dots = w_k = w_{k+1} = \dots.$$

Then it is obvious that  $\|T\| = w_1$  and

$$r(T) = \lim_{m \rightarrow \infty} \|T^m\|^{1/m} = w_2,$$

therefore  $T$  is not normaloid. By (2.5), we have  $T$  is  $(n, k)$ -quasi- $*$ -paranormal.

(5) An example of normaloid but not  $(n, 1)$ -quasi- $*$ -paranormal operator, for all non-negative integer  $n$ .

Let  $S$  be an operator on  $l_2(\mathbb{N})$  with the canonical orthogonal basis  $\{e_m\}_{m=1}^\infty$  defined by

$$Se_m = \begin{cases} e_{m+1} & \text{if } m = 2l - 1, l \in \mathbb{N}, \\ 0 & \text{if } m = 2l, l \in \mathbb{N}. \end{cases}$$

Put

$$T = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \text{ on } H = \mathbb{C}e_1 \oplus l_2(\mathbb{N}).$$

Then a simple calculation shows that  $T^m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  for all  $m \geq 2$  and  $T^*T = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}$  on  $H = \mathbb{C}e_1 \oplus l_2(\mathbb{N})$ , where  $P$  is the projection onto the closed subspace spanned by  $\{e_1, e_3, e_5, \dots\}$ . Hence  $T$  is normaloid because  $\|T^m\| = \|T\|^m = 1$ . We claim that  $T$  is not  $(n, 1)$ -quasi- $*$ -paranormal, for all nonnegative integer  $n$ . In fact, Lemma 2.2 shows that  $T$  is  $(n, 1)$ -quasi- $*$ -paranormal if and only if

$$T^*T^{*(1+n)}T^{1+n}T - (1+n)\mu^n T^*TT^*T + n\mu^{1+n}T^*T \geq 0$$

for all  $\mu > 0$ . Putting  $\mu = 1$ , we obtain

$$\begin{aligned} & T^*T^{*(1+n)}T^{1+n}T - (1+n)T^*TT^*T + nT^*T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - (1+n) \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} + n \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -P \end{pmatrix} < 0. \end{aligned}$$

Hence  $T$  is not  $(n, 1)$ -quasi- $*$ -paranormal.

### 3 A matrix representation

The following observation is a structure property for  $(n, k)$ -quasi- $*$ -paranormal operators.

**Observation 3.1.** *Suppose that  $T^k H$  is not dense. Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{T^k H} \oplus \ker(T^{*k})$$

where  $\overline{T^k H}$  is the closure of  $T^k H$ . If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal, then  $T_1$  is  $n$ - $*$ -paranormal,  $T_3^k = 0$  and  $\sigma(T) = \sigma(T_1) \cup \{0\}$ .

*Proof.* Since  $T_1^{1+n}z = T_1^{1+n}z$  for all  $z \in \overline{T^k H}$ ,  $T_1$  is  $n$ -\*-paranormal. Let  $x \in \ker(T^{*k})$ . Then

$$T^k x = \begin{pmatrix} T_1^k & \sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} \\ 0 & T_3^k \end{pmatrix} (0 \oplus x) \in \overline{T^k H}.$$

Hence  $T_3^k = 0$  and  $\sigma(T) = \sigma(T_1) \cup \{0\}$ .  $\square$

The above matrix representation of  $(n, k)$ -quasi- $*$ -paranormal operators motivates the following

**Question 3.2.** *Let  $H, K$  be two infinite dimensional complex Hilbert spaces. If  $A$  is  $n$ -\*-paranormal and  $C^k = 0$ , then the operator matrix*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

*acting on  $H \oplus K$  is  $(n, k)$ -quasi- $*$ -paranormal?*

Before giving a negative answer to this question, we present the following

**Theorem 3.3.** *Let  $T$  be an operator on  $H \oplus K$  defined as*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

*If  $A$  is  $n$ -\*-paranormal and surjective and  $C^k = 0$ , then  $T$  is similar to an  $(n, k)$ -quasi- $*$ -paranormal operator.*

*Proof.* Since  $A$  is surjective and  $C^k = 0$ , we have  $\sigma_s(A) \cap \sigma_a(C) = \emptyset$ , where  $\sigma_s(\cdot)$  and  $\sigma_a(\cdot)$  denote the surjective spectrum and the approximative point spectrum respectively. It then follows from part (c) of Theorem 3.5.1 in [15] that there exist some  $S \in L(K, H)$  for which  $AS - SC = B$ . Since

$$\begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},$$

it follows that  $T$  is similar to  $R := \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ . Let  $x = x_1 \oplus x_2 \in H \oplus K$ . Since  $A$  is  $n$ -\*-paranormal and  $C^k = 0$ , we have

$$\begin{aligned} \|R^*(R^k x)\| &= \|R^*(R^k(x_1 \oplus x_2))\| = \|A^*(A^k x_1)\| \\ &\leq \|A^{1+n}(A^k x_1)\|^{1/(1+n)} \|A^k x_1\|^{n/(1+n)} \\ &= \|R^{1+n}(R^k(x_1 \oplus x_2))\|^{1/(1+n)} \|R^k(x_1 \oplus x_2)\|^{n/(1+n)} \\ &= \|R^{1+n}(R^k x)\|^{1/(1+n)} \|R^k x\|^{n/(1+n)}. \end{aligned}$$

Thus  $T$  is similar to an  $(n, k)$ -quasi- $*$ -paranormal operator.  $\square$

The following simple example provides a negative answer to Question 3.2 and a question posed by Mecheri [18]: Is the operator matrix

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

acting on  $H \oplus K$  is  $k$ -quasi- $*$ -class  $A$  if  $A$  is  $*$ -class  $A$  and  $C^k = 0$ ? It also shows us that [18, Theorem 2.1] is not correct.

**Example 3.4.** Let  $l_2(\mathbb{N})$  with the canonical orthogonal basis  $\{e_m\}_{m=1}^\infty$  and put

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \text{ on } H = l_2(\mathbb{N}) \oplus l_2(\mathbb{N}).$$

Obviously,  $A = 0$  is  $n$ - $*$ -paranormal (respectively,  $*$ -class  $A$ ) for all nonnegative integer  $n$  and  $C^k = 0^k = 0$ , since we know that  $k \geq 1$  from the assumption. However,  $T$  is not  $(n, k)$ -quasi- $*$ -paranormal for all nonnegative integer  $n$ , since

$$\begin{aligned} \|T^*(e_1 \oplus 0)\| &= \left\| \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} (e_1 \oplus 0) \right\| = \|0 \oplus e_1\| \\ &= 1 > 0 = \|T^{1+n+k}(e_1 \oplus 0)\|^{1/(1+n)} \|T^k(e_1 \oplus 0)\|^{n/(1+n)}. \end{aligned}$$

In particular,  $T$  is not  $(1, k)$ -quasi- $*$ -paranormal. Hence by Theorem 2.1(1),  $T$  is not  $k$ -quasi- $*$ -class  $A$ .

## 4 Joint (approximate) point spectrum and SVEP

[23, Theorem] illustrated that the following result is not true even for paranormal operators.

**Theorem 4.1.** *Let  $T$  be  $(n, k)$ -quasi- $*$ -paranormal and  $0 \neq \lambda \in \mathbb{C}$ .*

(1) *If  $(T - \lambda)x = 0$ , then  $(T^* - \bar{\lambda})x = 0$ . Consequently*

$$\ker(T - \lambda) \subseteq \ker(T^* - \bar{\lambda}).$$

(2) *If  $(T - \lambda)x_m \rightarrow 0$  for a sequence  $\{x_m\}_{m=1}^\infty$  of unit vectors, then  $(T^* - \bar{\lambda})x_m \rightarrow 0$ .*

Clearly, if  $\lambda = 0$ , then the above properties hold for  $(n, 0)$ -quasi- $*$ -paranormal (that is  $n$ - $*$ -paranormal) operators. However [22, Example 6] showed that, when  $\lambda = 0$ , the above properties do not hold even for quasi-hyponormal (that is,  $(0, 1)$ -quasi- $*$ -paranormal) operators.

*Proof.* (1) We may assume that  $\lambda \in \sigma_p(T)$ , where  $\sigma_p(T)$  is the point spectrum of  $T$ . Let  $x \in \ker(T - \lambda)$  and  $\|x\| = 1$ . Then

$$\|T^*(T^k x)\|^{n+1} \leq \|T^{1+n}(T^k x)\| \|T^k x\|^n,$$



hence

$$|\lambda|^{k(n+1)} \|T^*x\|^{n+1} \leq |\lambda|^{k(n+1)} |\lambda|^{n+1}.$$

That is,

$$\|T^*x\| \leq |\lambda|.$$

Thus we have

$$\begin{aligned} \|T^*x - \bar{\lambda}x\|^2 &= (T^*x - \bar{\lambda}x, T^*x - \bar{\lambda}x) \\ &= (T^*x, T^*x) - \bar{\lambda}(x, T^*x) - \lambda(T^*x, x) + |\lambda|^2 \\ &= \|T^*x\|^2 - \bar{\lambda}(Tx, x) - \lambda(x, Tx) + |\lambda|^2 \\ &\leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0. \end{aligned}$$

Hence  $\|T^*x - \bar{\lambda}x\|^2 \leq 0$  and consequently  $x \in \ker(T^* - \bar{\lambda})$ .

(2) Let  $(T - \lambda)x_m \rightarrow 0$  for unit vectors  $\{x_m\}$  and let  $l \in \mathbb{N}$ . Since

$$T^l = (T - \lambda + \lambda)^l = \sum_{j=1}^l \binom{l}{j} \lambda^{l-j} (T - \lambda)^j + \lambda^l,$$

we have  $(T^l - \lambda^l)x_m \rightarrow 0$ . It then follows from

$$\begin{aligned} \left| \|\lambda^l x_m\| - \|(T^l - \lambda^l)x_m\| \right| &\leq \|T^l x_m\| = \|\lambda^l x_m + (T^l - \lambda^l)x_m\| \\ &\leq \|\lambda^l x_m\| + \|(T^l - \lambda^l)x_m\| \end{aligned}$$

that  $\|T^l x_m\| \rightarrow |\lambda|^l$ . In particular, we have

$$\|T^{1+n+k} x_m\| \rightarrow |\lambda|^{1+n+k} \text{ and } \|T^k x_m\| \rightarrow |\lambda|^k. \quad (4.1)$$

Moreover,

$$\left| \|T^* \lambda^k x_m\| - \|T^*(T^k - \lambda^k)x_m\| \right| \leq \|T^* T^k x_m\|. \quad (4.2)$$

Since  $T$  is  $(n, k)$ -quasi- $*$ -paranormal,

$$\|T^*(T^k x_m)\|^{n+1} \leq \|T^{1+n}(T^k x_m)\| \|T^k x_m\|^n.$$

Then it follows from (4.1) and (4.2) that

$$\limsup_{m \rightarrow \infty} \|T^* x_m\| \leq |\lambda|.$$

Since

$$\begin{aligned} \|T^* x_m - \bar{\lambda}x_m\|^2 &= (T^* x_m - \bar{\lambda}x_m, T^* x_m - \bar{\lambda}x_m) \\ &= (T^* x_m, T^* x_m) - \bar{\lambda}(x_m, T^* x_m) - \lambda(T^* x_m, x_m) + |\lambda|^2 \\ &= \|T^* x_m\|^2 - \bar{\lambda}(Tx_m, x_m) - \lambda(x_m, Tx_m) + |\lambda|^2 \\ &= \|T^* x_m\|^2 - \bar{\lambda}((T - \lambda)x_m, x_m) - \lambda(x_m, (T - \lambda)x_m) - |\lambda|^2, \end{aligned}$$

we have

$$\limsup_{m \rightarrow \infty} \|T^* x_m - \bar{\lambda}x_m\|^2 \leq |\lambda|^2 - |\lambda|^2 = 0.$$

This establishes that  $(T^* - \bar{\lambda})x_m \rightarrow 0$ . □

**Remark 4.2.** We note that, with the Berberian faithful  $*$ -representation [6] at hand, part (2) of Theorem 4.1 can also be deduced from part (1) of it.

For  $T \in L(H)$ , let  $\sigma_p(T)$ ,  $\sigma_{jp}(T)$ ,  $\sigma_a(T)$  and  $\sigma_{ja}(T)$  denote the point spectrum, joint point spectrum, approximate point spectrum and joint approximate point spectrum of  $T$ , respectively (see [2]). Many mathematicians shown that, for some nonhyponormal operators  $T$ , the nonzero points of its point spectrum and joint point spectrum are identical, the nonzero points of its approximate point spectrum and joint approximate point spectrum are identical (see [2, 24, 25, 26]). The following corollary, which is an immediate consequence of Theorem 4.1, extends this result to the class of  $(n, k)$ -quasi- $*$ -paranormal operators.

**Corollary 4.3.** *If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal, then*

$$\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\} \text{ and } \sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}.$$

In the next example, we invoke Uchiyama's example again to illustrate that the above equalities do not hold even for paranormal operators.

**Example 4.4.** Recall that Uchiyama have constructed in [23, Theorem] an operator defined by

$$T = \begin{pmatrix} 1 & \alpha e_1 \otimes e_1 \\ 0 & U + 1 \end{pmatrix} \text{ on } H = \mathbb{C}e_1 \oplus l_2(\mathbb{N}),$$

where  $U$  is the unilateral right shift operator on  $l_2(\mathbb{N})$  with the canonical orthogonal basis  $\{e_n\}_{n=1}^\infty$ . He proved that  $T$  is paranormal for  $0 < \alpha < \frac{1}{4}$  and  $\ker(T - 1) = \mathbb{C}e_1 \oplus \{0\}$ . A step further, we have  $\ker(T^* - 1) = \{ae_1 \oplus (be_1 + a\alpha e_2) : a, b \in \mathbb{C}\}$ , hence

$$\ker(T - 1) \cap \ker(T^* - 1) = \{0\}.$$

Consequently,  $1 \in \sigma_p(T) \setminus \sigma_{jp}(T)$ . Evidently,  $1 \in \sigma_a(T)$ . Next, we show that  $1 \notin \sigma_{ja}(T)$ . Otherwise, there exists a sequence  $\{x_n\}_{n=1}^\infty$  of unit vectors satisfying  $(T - 1)x_n \rightarrow 0$  and  $(T^* - 1)x_n \rightarrow 0$ . For  $n \in \mathbb{N}$ , let  $x_n = a_n e_1 \oplus (b_{1,n}, b_{2,n}, \dots) \in \mathbb{C}e_1 \oplus l_2(\mathbb{N})$ . Then

$$a_n^2 + \sum_{k=1}^\infty b_{k,n}^2 = 1, \tag{4.3}$$

$$\alpha b_{1,n}^2 + \sum_{k=1}^\infty b_{k,n}^2 \rightarrow 0, \tag{4.4}$$

and

$$(\alpha a_n + b_{2,n})^2 + \sum_{k=3}^\infty b_{k,n}^2 \rightarrow 0. \tag{4.5}$$

By (4.4), we have  $\sum_{k=1}^\infty b_{k,n}^2 \rightarrow 0$ ,  $\sum_{k=2}^\infty b_{k,n}^2 \rightarrow 0$  and  $b_{2,n} \rightarrow 0$ . Then by (4.3), we have  $a_n^2 \rightarrow 1$ .

Thus, we have  $(\alpha a_n + b_{2,n})^2 + \sum_{k=3}^\infty b_{k,n}^2 = \alpha^2 a_n^2 + 2\alpha a_n b_{2,n} + \sum_{k=2}^\infty b_{k,n}^2 \rightarrow \alpha^2$ , which contradicts to (4.5).

**Corollary 4.5.** *If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal and  $\lambda \neq \mu$ , then*

$$\ker(T - \lambda) \perp \ker(T - \mu).$$

*Proof.* Without loss of generality, we may suppose that  $\mu \neq 0$ . Let  $x \in \ker(T - \lambda)$  and  $y \in \ker(T - \mu)$ . Then by Theorem 4.1(1), we have

$$\lambda(x, y) = (Tx, y) = (x, T^*y) = (x, \overline{\mu}y) = \mu(x, y),$$

which implies  $(x, y) = 0$  and so  $\ker(T - \lambda) \perp \ker(T - \mu)$ .  $\square$

**Corollary 4.6.** *If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal, then  $\ker(T^{1+k}) = \ker(T^{2+k})$  and  $\ker(T - \lambda) = \ker(T - \lambda)^2$  when  $0 \neq \lambda \in \mathbb{C}$ .*

*Proof.* Since  $T$  is  $(n, k)$ -quasi- $*$ -paranormal, it is  $(n, k + 1)$ -quasi- $*$ -paranormal by Theorem 2.1(4), and hence it is  $(n + 1, k)$ -quasiparanormal by Theorem 2.1(2). Therefore  $\ker(T^{n+k+2}) = \ker(T^{1+k})$  and so  $\ker(T^{1+k}) = \ker(T^{2+k})$ . When  $0 \neq \lambda \in \mathbb{C}$ , we need only to show that  $\ker(T - \lambda) \supseteq \ker(T - \lambda)^2$ , since  $\ker(T - \lambda) \subseteq \ker(T - \lambda)^2$  is clear. Theorem 4.1(1) implies that  $(T - \lambda)^*(T - \lambda)\ker(T - \lambda)^2 = \{0\}$ , hence  $(T - \lambda)\ker(T - \lambda)^2 \subseteq \ker(T - \lambda)^* \cap (T - \lambda)H = \{0\}$  and so  $\ker(T - \lambda)^2 \subseteq \ker(T - \lambda)$ .  $\square$

An operator  $T \in L(H)$  is said to have single valued extension property at  $\lambda_0 \in \mathbb{C}$  (SVEP at  $\lambda_0$  for brevity) if for every open neighborhood  $\mathcal{U}$  of  $\lambda_0$ , the only analytic function  $f : \mathcal{U} \rightarrow H$  which satisfies the equation  $(\lambda I - T)f(\lambda) = 0$  for all  $\lambda \in \mathcal{U}$  is the constant function  $f \equiv 0$ . Let  $\mathcal{S}(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$ . An operator  $T \in L(H)$  is said to have SVEP if  $\mathcal{S}(T) = \emptyset$ .

**Corollary 4.7.** *If  $T$  is  $(n, k)$ -quasi- $*$ -paranormal, then  $T$  has SVEP.*

*Proof.* It follows directly from Corollary 4.5 and [27, Lemma 3.5] (or, Corollary 4.6 and [14, Proposition 1.8]).  $\square$

SVEP, which is an important property in local spectral theory and Fredholm theory, has a number of consequences. We list in the next corollary some of these.

For  $T \in L(H)$ , let  $\sigma_W(T)$ ,  $\sigma_{SF_+^-}(T)$ ,  $\sigma_{BW}(T)$ ,  $\sigma_{SBF_+^-}(T)$  and  $\sigma_{LD}(T)$  denote the Weyl spectrum, upper semi-Weyl spectrum, B-Weyl spectrum, upper semi-B-Weyl spectrum and left Drazin spectrum of  $T$ , respectively (see [7, 28]).

We say that  $T \in L(H)$  is algebraically  $(n, k)$ -quasi- $*$ -paranormal, if there exist a non-constant complex polynomial  $p$  such that  $p(T)$  is  $(n, k)$ -quasi- $*$ -paranormal. Let  $H(\sigma(T))$  denote the space of all functions analytic on some open neighborhood  $\mathcal{U}$  containing  $\sigma(T)$ . If  $T$  has SVEP then so does  $f(T)$  for all  $f \in H(\sigma(T))$ ; conversely, if  $p(T)$  has SVEP for some nonconstant polynomial  $p$  then  $T$  does ([1, Theorem 2.40]). Hence algebraically  $(n, k)$ -quasi- $*$ -paranormal operators have SVEP by Corollary 4.7.

**Corollary 4.8.** *Let  $f \in H(\sigma(T))$ . If  $T$  or  $T^*$  is algebraically  $(n, k)$ -quasi- $*$ -paranormal, then*

- (1)  $f(T)$  and  $f(T^*)$  obey to a-Browder's theorem.
- (2)  $f(T)$  possesses property (gb), when  $T^*$  is algebraically  $(n, k)$ -quasi- $*$ -paranormal.

(3)  $\sigma_W(f(T)) = f(\sigma_W(T))$  and analogous equality holds for  $\sigma_{SF_+^-}(\cdot)$ .

(4) if  $f$  is nonconstant on each component of open neighborhood  $\mathcal{U}$  containing  $\sigma(T)$ , then  $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$  and analogous equality holds for  $\sigma_{SBF_+^-}(\cdot)$ .

*Proof.* (1), (2) and (3) follow from [1, Corollary 3.73], [28, Theorem 2.14] and [1, Corollary 3.72], respectively.

(4) By [3, Theorem 2.4], it remains to show that  $\sigma_{SBF_+^-}(f(T)) = f(\sigma_{SBF_+^-}(T))$ . From [3, Theorem 2.1] and [28, Lemma 2.3], it follows that  $\sigma_{SBF_+^-}(T) \cup (\mathcal{S}(T) \cap \mathcal{S}(T^*)) = \sigma_{LD}(T)$  for any  $T \in L(H)$ . And then the assertion follows from the fact that left Drazin spectrum satisfies the spectral mapping theorem for such a function  $f$ .  $\square$

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